Modelling of dependence between critical failure and preventive maintenance: The repair alert model

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Abstract

We consider the competing risks problem for a repairable unit which at each sojourn may be subject to either a critical failure, or a preventive maintenance (PM) action. It is reasonable to expect a dependence between the failure mechanism and the PM regime. In the talk we present and study a new model, called the repair alert model, for handling such cases. This model is a special case of random signs censoring, which was introduced by Cooke (1993). The talk summarizes ideas and results from the paper Lindqvist, Støve and Langseth (2004).

1 Introduction

Consider a component which is subject to failure at a random time X. Assume that the failure can be avoided by a possible preventive maintenance (PM) at some random time Z. If $Z < X$, then we will not observe the failure but rather the PM event. On the other hand, if $X < Z$, then the failure is experienced. The situation is thus a case of competing risks, where in effect we observe only the pair (Y, δ), where $Y = \min(X, Z)$ and $\delta = I(Z < X)$ is the indicator of the event $\{Z < X\}$.

During operation, a (competent) maintenance crew is likely to have some information regarding the state of the component. The crew will use this insight to perform maintenance in order to avoid typically more costly component failures. Hence it is reasonable to expect a dependence between the time X of failure of the component and the time Z of PM actions. We are therefore faced with *dependent competing risks.* Cooke (1993, 1996) introduced the notion of *random signs censoring* which is tailored for such cases. In our notation, random signs censoring can be defined as follows:

Definition 1 *Let (X, Z) be a pair of life variables. Then Z is called a random signs censoring of X if the event $\{Z < X\}$ is stochastically independent of X.*

Thus, random signs censoring means that the event that the failure of the component is preceded by PM, is not influenced by the time X at which the component fails or would have failed without PM.

It is well known that the marginal distributions of X and Z are not identifiable from the observed (Y, δ) without making assumptions on the dependence between them. Cooke's random signs censoring makes such assumptions, leading to identifiability of the distribution of X.

In the repair alert model, the PM time Z is a random signs censoring of X, but additional structure is imposed. The clue is to introduce the so called repair alert function, which describes the "alertness" of the maintenance crew as a function of time. The repair alert model generalizes the model of Langseth and Lindqvist (2003), who used the failure rate function itself as the measure of alertness.

The main result below (Theorem 1) implies that whenever there exists a model for (X, Z) satisfying the random signs requirements, there is a unique repair alert model having the same distribution of the observable (Y, δ).

2 Notation

We shall assume that (X, Z) is a pair of continuously distributed life variables, with the property that $P(X = Z) = 0$. The following notation appears to be standard.

Let $F_X(t) = P(X \leq t)$ and $F_Z(t) = P(Z \leq t)$ be the cumulative distribution functions of X and Z, respectively. Next, the *subdistribution functions* of X and Z are defined as, respectively, $F_X^*(t) = P(X \leq t, X < Z)$ and $F_Z^*(t) = P(Z \leq t, Z < X)$.

Note that the functions F_X^* and F_Z^* are nondecreasing with $F_X^*(0) = 0$ and $F_Z^*(0) = 0$. Moreover, we have $F_X^*(\infty) + F_Z^*(\infty) = 1$. Any pair of functions K_1, K_2 satisfying these conditions, will be referred to as a *subdistribution pair.*

We will also use the notion of *conditional distribution functions*, defined by $\tilde{F}_X(t) = P(X \leq t | X < Z)$ and $\tilde{F}_Z(t) = P(Z \leq t | Z < X)$. Note then that $\tilde{F}_X(t) = F_X^*(t)/F_X^*(\infty)$, $\tilde{F}_Z(t) = F_Z^*(t)/F_Z^*(\infty)$.

For convenience we assume the existence of densities corresponding to each of the functions defined above, i.e. $f_X(t) = F_X'(t)$, $f_X^*(t) = F_X^{*'}(t)$, $\tilde{f}_X(t) = \tilde{F}_X'(t)$, and similarly for Z.

3 The repair alert model

Definition 2 *The pair (X, Z) of life variables satisfies the requirements of the repair alert model provided the following two conditions both hold:*

(i) *Z is a random signs censoring of X*

(ii) *There exists an increasing function G with $G(0) = 0$ such that for all $x > 0$,*

$$P(Z \leq z | Z < X, X = x) = \frac{G(z)}{G(x)}, \quad 0 < z \leq x \, .$$

The function G is called *the cumulative repair alert function.* Its derivative g (which we shall assume exists) is called *the repair alert function.*

The repair alert model is hence a specialization of random signs censoring, obtained by introducing the repair alert function G.

Part (ii) of the definition can be described as follows. Given that there would be a failure at time $X = x$, and given that the maintenance crew will perform a PM before that time (i.e. given that $Z < X$), the conditional density of the time Z of this PM is proportional to the repair alert function g. The repair alert function is meant to reflect the reaction of the maintenance crew. More precisely, $g(t)$ ought to be high at times t for which failures are expected and the alert therefore should be high. As mentioned above, Langseth and Lindqvist (2003) simply put $g(t) = \lambda(t)$ where $\lambda(t)$ is the failure rate of the marginal distribution of X.

It can be seen that the repair alert model is completely determined by the marginal distribution function F_X, the (cumulative) repair alert function G, the probability $q \equiv P(Z < X)$, and the assumption that X is independent of the event $\{Z < X\}$ (i.e. random signs censoring).

Theorem 1 below is our main result. Here the equivalence between (ii) and (iii) rephrases the main result of Cooke (1993). Now from the equivalence of these conditions to condition (i) it follows that if a random signs censoring exists with a given set of subdistribution functions, then this can be taken to be a repair alert model. Moreover, the theorem states that while there may exist several random signs censorings with given subdistribution functions, there is a unique repair alert model. Finally, note that condition (iii) means that the conditional distribution of X strictly dominates the conditional distribution of Z.

Theorem 1 *Let K_1, K_2 be a subdistribution pair such that K_2 is differentiable. Then the following are equivalent*

(i) *There exists a pair (X, Z) of life variables which satisfy the requirements of the repair alert model and which are such that*

$$F_X^*(t) = K_1(t) \text{ for all } t \geq 0, \; F_Z^*(t) = K_2(t) \text{ for all } t \geq 0$$

(ii) *There exists a pair (X, Z) of life variables such that Z is a random signs censoring of X and which are such that*

$$F_X^*(t) = K_1(t) \text{ for all } t \geq 0, \; F_Z^*(t) = K_2(t) \text{ for all } t \geq 0$$

(iii)

$$\frac{K_1(t)}{K_1(\infty)} < \frac{K_2(t)}{K_2(\infty)} \text{ for all } t > 0$$

Moreover, if condition (iii) holds and (X, Z) has the repair alert model in (i), then the cumulative repair alert function G is uniquely (modulo a multiplicative constant) given by

$$G(t) = \exp\left\{\int_{t_0}^{t} \frac{\tilde{f}_Z(w)}{\tilde{F}_Z(w) - F_X(w)} dw\right\} \tag{1}$$

for all $t > 0$, where $t_0 > 0$ is a fixed, arbitrary constant.

4 Some properties of the repair alert model

In this section we discuss some implications of the repair alert model. In order to help intuition, we sometimes consider the power version $G(t) = t^\beta$ where $\beta > 0$ is a parameter. Then $g(t) = \beta t^{\beta-1}$ so $\beta = 1$ means a constant repair alert function, while $\beta < 1$ and $\beta > 1$ correspond to, respectively, a decreasing and increasing repair alert function.

It can be shown that

$$E(Z|Z < X) = \int_0^\infty (1 - \tilde{F}_Z(z))dz = E(X) - E\left[\frac{M(X)}{G(X)}\right]$$

where $M(x) = \int_0^x G(t)dt$. For the special case when $G(t) = t^\beta$, we obtain the simple result

$$E(Z|Z < X) = \frac{\beta}{\beta + 1} E(X)$$

which clearly indicates that good PM performance corresponds to large values of β.

Instead of merely considering the conditional expectation $E(Z|Z < X)$ one may more generally study the conditional *distribution* of Z given $Z < X$. Part (i) of Theorem 2 below gives a result on the behaviour of Z in relation to X for different repair alert functions.

An alternative way of considering the relation between Z and X is via the distribution of the remaining time to the potential failure, given that a PM is performed at time $Z = z$. More specifically, this is the conditional distribution of $X - Z$ given $Z = z,\ Z < X$. Intuitively, a good PM performance would mean that this distribution is small (stochastically). Part (ii) of Theorem 2 gives a result on the behaviour of this distribution for different repair alert functions.

Theorem 2 *Suppose (X, Z) has a repair alert distribution with fixed parameters q and F_X, while the cumulative repair alert function is either $G^{(1)}$ or $G^{(2)}$. Suppose further that $G^{(1)}(t)/G^{(2)}(t)$ is an increasing function of t. Then*

(i) *The conditional distribution of Z given $Z < X$ under $G^{(1)}$ stochastically dominates the corresponding distribution under $G^{(2)}$.*

(ii) *The conditonal distribution of $W = X - Z$ given $Z = z, Z < X$ under $G^{(1)}$ is stochastically dominated by the corresponding distribution under $G^{(2)}$.*

Theorem 2 immediately implies that if $G(t) = t^\beta$, then (i) the conditional distribution of Z given $Z < X$ is stochastically increasing in β, and (ii) the conditional distribution of $X - Z$ given $Z = z, Z < X$ is stochastically decreasing in β.

Consider next $Y = \min(X, Z)$, which is the actual operation time of the component. The following result may shed some light on the influence of the parameters of the repair alert model. Let $G(t) = t^\beta$. Then

$$E(Y) = E(X)\left(1 - \frac{q}{\beta + 1}\right).$$

5 Statistical inference in the repair alert model

Let $(y_1, \delta_1), (y_2, \delta_2), \ldots, (y_N, \delta_N)$ be N i.i.d. observations of $(Y, \delta) \equiv (\min(X, Z),\ I(Z < X))$. The observations may more conveniently be represented as $x_1, ..., x_m$ and $z_1, ..., z_n$, which are, respectively, the observed times to failure and the observed times for PM.

Since $q = P(Z < X)$, a natural estimator is $\hat{q} = n/N$. Since F_X under the repair alert model equals the conditional distribution function $\tilde{F}_X$, the natural estimator $\hat{F}_X$ of F_X is the empirical distribution function based solely on the $x_1, ..., x_m$. Similarly, $\tilde{F}_Z$ can be estimated by the empirical distribution function $\hat{\tilde{F}}_X(t)$ based on $z_1, ..., z_n$. Rewriting (1) as

$$G(t) = \exp\{\int_{\tilde{F}_Z(t_0)}^{\tilde{F}_Z(t)} \frac{dy}{y - F_X(\tilde{F}_Z^{-1}(y))}\}$$

we thus obtain a simple nonparametric plug-in estimator of $G(t)$ by replacing F_X and $\tilde{F}_Z$ by their empirical versions.

A simple but perhaps useful parametric model is obtained by letting X be exponentially distributed with density $f_X(x) = \lambda e^{-\lambda x}$, and letting $G(t) = t^\beta$. In this case λ, β, q are the parameters of the repair alert model. As shown in Lindqvist et al. (2004), the log likelihood is given by

$$\begin{aligned} l(\lambda, \beta, q) &= m \log(1-q) + n \log q + (n+m) \log \lambda + n \log \beta - \lambda \sum_{i=1}^{m} x_i \\ &+ \sum_{i=1}^{n} (\beta - 1) \log(\lambda z_i) + \sum_{i=1}^{n} \log(\int_{\lambda z_i}^{\infty} w^{-\beta} e^{-w} dw) \end{aligned}$$

so the maximum likelihood estimates of the parameters can be found by maximizing this expression. Alternatively, one may use the EM-algorithm to obtain rather simple iterative formulae for the parameter estimates (Lindqvist et al, 2004).

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